Indian Statistical Institute
B.Math I Year
Second Semester Examination, 2005-2006
Probability Theory II

Time: 3 hrs | Date:03-05-06 | Max. Marks : 100

Note: The paper carries 112 marks. Any score above 100 will be treated as 100.

1. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each having an exponential distribution with parameter $\lambda > 0$. Define $Y_k = \sum\limits_{j=1}^{k} X_j,\ \ k = 1, 2, \ldots, n$.

   a) Find the joint probability density function of $Y_1, Y_2, \ldots, Y_n$.

   b) What is the marginal distribution of Y_k for $1 \leq k \leq n$?

   c) Taking $n = 2$, find the conditional probability density function of X_1 given $X_1 + X_2 = y$, for $y > 0$. [10+4+6]

2. Let $X_1, X_2, \ldots, X_n$ be random variables having finite second moments. Let $\sum = ((\sigma_{ij})),\ 1 \leq i, j \leq n$ be the real $(n \times n)$ matrix given by $\sigma_{ij} =$ Cov (X_i, X_j). Show that $\sum$ is a symmetric nonnegative definite matrix. [10]

3. Let $U_1, U_2, \ldots, U_n$ be independent absolutely continuous random variables with common distribution function F and common probability density function f. Assume $f(x) = F'(x)$ for all x. Let $X_1, X_2, \ldots, X_n$ denote the corresponding order statistics; that is X_k is the k-th smallest among U_i's.

   a) Show that the distribution function of X_k is given by

   $$F_{X_k}(x) = \sum_{j=k}^{n} \binom{n}{j} (F(x))^j (1 - F(x))^{n-j}, \quad x \in \mathbb{R}$$

   for any $1 \leq k \leq n$.

   b) Find the probability density function of $\min\{U_1, U_2, \ldots, U_n\}$. [12+5]

4. Let X have a uniform distribution over $(-1, 1)$. Find its characteristic function ϕ_X, and show that ϕ_X is continuous. [10]

5. For $n = 1, 2, \ldots$ let the random variable X_n have $N(0, \frac{1}{n})$ distribution. Let X be a random variable such that $P(X = 0) = 1$. Assume that all X_n and X are defined on the same probability space.

   a) Show that $\{X_n\}$ converges to X in distribution as $n \to \infty$.

   b) Does $\{X_n\}$ converge to X in probability ? [10+5]

6. 200 numbers are rounded off to the nearest integer and then added. Assume the individual round-off errors are independent and uniformly distributed over $(-\frac{1}{2}, \frac{1}{2})$. Find the probability that the computed sum will differ from the sum of the original 200 numbers by more than 5.

   [20]

7. Let χ_n^2 denote a random variable having $\chi^2(n)$ distribution. Using the central limit theorem, find $\lim_{n\to\infty} P(\chi_n^2 \leq (\sqrt{2n}\ y\ +\ n))$ where $y > 0$.

   [20]